J. Fluid Mech. (1970). vol. 44, part 1, pp. 113–127
Printed in Great Britain

The calculation of inviscid hypersonic flow past the lower surface of a delta wing

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(Received 19 December 1969)

Kennett (1963) calculated the hypersonic flow fields past the lower (compression) surface of a delta wing, using the one-strip approximation of the method of integral relations. He obtained solutions only for wings with detached shocks. In this paper, his solutions are extended to wings with attached shocks. Here, the sonic point is inboard of the leading edge which makes the problem mixed. The solutions compare very well with the numerical solutions of the full equations by Babaev (1963*a*) both in the shock shapes and pressure distributions for various Mach numbers.

1. Introduction

When hypersonic flow takes place past the lower surface of a delta wing, a shock wave is formed upstream of the lower surface. This shock may be attached or detached from the leading edges of the wing, depending on the Mach number, angle of attack and sweepback angle of the wing.

The flow for the detached-shock case has been calculated by a number of workers but for some applications, as Kuchemann (1964) has remarked, it is the attached-shock case that is more important since it is the one that is relevant to the design of proposed commercial hypersonic aircraft. It is the attached shock case that we shall be concerned with in this paper.

In the early fifties, Maslen (1952) and Fowell (1956) used the linearization method to obtain some calculations. Their methods are inapplicable for flows with shock waves of more than negligible strength when linearization breaks down for perturbations are then no longer small.

Two approaches have been made to date to the problem of calculating the flow past a delta wing with attached-shock wave of arbitrary strength. The first and more important, since it sets a standard which will be aimed at in the calculations which follow was by the Russian worker Babaev (1963*a*, *b*), who in two papers described and gave the results of calculations for delta wings at Mach numbers 4, 6 and (effectively) infinity. He replaced the original set of partial differential equations by their finite difference equivalents. After assuming some shape of the shock wave, he used the steepest descent method to solve the finite difference equations. He found a more precise shape of the shock and with this he solved the system of equations again and refined the shock shape still further. The process was continued until the last two shock shapes were identical. The published account of his work suggests that the method required considerable development and computer time, and notwithstanding its success has not been taken up since.

The second approach has been made by Squire (1967) and B. A. Woods (private communication) on the basis of a thin shock layer approximation introduced by Messiter (1963). Their work is thus restricted to hypersonic flow only. Squire's results entail a further approximation in obtaining a smooth solution to Messiter's equations. To satisfy the boundary conditions he is forced to relax the flow tangency condition on part of the wing surface, so that his calculations in effect are for a delta wing which is slightly non-planar over some of its span.

Woods's solutions satisfy the boundary conditions exactly but contain discontinuities. Clearly the thin shock-layer approximation leads to the introduction of spurious features in the flow. Neither workers' results are better than fair as regards pressure distribution and overall force, so that this approximation has limitations both in principle and in practice.

The detached shock case has been dealt with by a number of people. Messiter (1963) and Hida (1965) based their calculations on Newtonian assumptions and thin shock-layer approximation. Squire has also improved on their work. Kennet (1963) used the method of integral relations to obtain his solutions. After reducing the governing partial differential equations to ordinary differential equations, and replacing one of the momentum equations by an equation expressing the constancy of entropy on the wing surface, he started his integration from the centreline outwards with a value of the shock distance which he obtained by requiring that the sonic singular point occurred at the leading edge.

For an attached shock case, we shall follow Kennet in the derivation of the equations of motion and the reduction to ordinary differential equations. But, instead of starting the numerical integration from the centreline, we shall start from the sonic singular point.

It is of interest to remark here that Brook (1965) attempted the attached shock case using precisely the same equations as Kennet, and integrating from the centreline outwards. He was, however, unable to obtain a solution which matched the uniform solution at the Mach-cone singular point and therefore he concluded that the extension of the method of integral relations with one strip approximation to the mixed cross-flow problem leads to an indeterminate solution. We shall see later on that the problem is indeed determinate.

We now analyse the problem. The flow is conical with centre of conicity at the apex of the wing. We shall consider only the one-strip approximation of the method of integral relations, assuming that the flow fields vary across the shock layer according to an assumed law.

The flow is uniform in the region between the leading edge and the sonic point (ADB in figure 1). This region is bounded by the Mach cone BD, the plane shock BA and the wing surface AD.

The flow is non-uniform in the centre region of the wing. At the point B, is the point of intersection of the Mach cone BD and the shock wave as the latter begins to curve. Between the point B and the plane of symmetry of the wing, the shape of the shock wave is unknown and must be determined from the system of equations.

At the leading edge A, the shock distance is zero. Here a cubic is solved to give the shock slope, and with this known, the other flow fields are determined. The sonic point D is found by noting that at this point, the velocity of sound is equal to the azimuthal velocity.

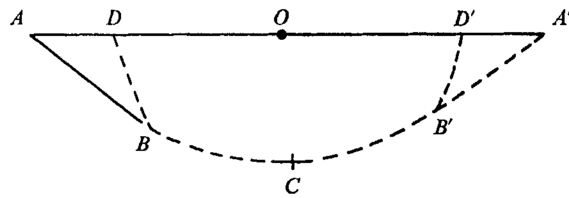


FIGURE 1. Section across shock layer.

The boundary condition at O as we shall see is that both the azimuthal velocity and shock slope vanish there (the former from symmetry and the latter from the fact that the shock is parallel to the wing surface there).

Having found our flow fields at the sonic point and with the boundary condition at the root chord in mind, the resulting boundary-value problem is solved numerically using the fourth-order Runge–Kutta–Merson method.

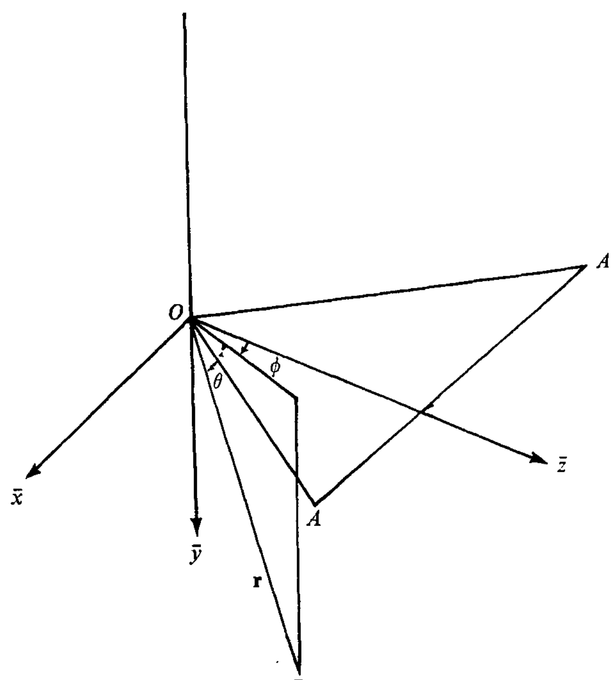


FIGURE 2. Diagram to show co-ordinate systems.

2. Formulation of governing equations

Consider a flat delta wing lying in the $\bar{x}\bar{z}$ plane (figure 2). The origin of the Cartesian co-ordinate system is at the vertex of the wing, the $\bar{z}$ axis is pointing downstream along the wing axis and the $\bar{y}$ axis is vertically downwards. The upstream velocity vector U_∞ lies in the $\bar{x} = 0$ plane and is inclined at angle α to the $\bar{y} = 0$ plane. It is convenient to introduce a spherical polar co-ordinate system $(\bar{r}, \bar{\theta}, \bar{\phi})$. Let the velocity components in this system be $\bar{q}_r$, $\bar{q}_\theta$, $\bar{q}_\phi$ and let the pressure, density, temperature, enthalpy, and time be $\bar{p}, \bar{\rho}, \bar{T}, \bar{h}, t$, respectively. We now introduce the dimensionless dependent variables $p = \bar{p}/P_{\text{ref}}$, $\rho = \bar{\rho}/\rho_{\text{ref}}$, $\bar{u} = \bar{q}_r/R^{\frac{1}{2}}T_{\text{ref}}$, $\bar{v} = \bar{q}_\theta/R^{\frac{1}{2}}T_{\text{ref}}$, $\bar{w} = \bar{q}_\phi/R^{\frac{1}{2}}T_{\text{ref}}$, where R is gas constant and the ref quantities are defined later. It is advantageous to replace $(\bar{\theta}, \bar{\phi})$ by (θ, ϕ) where $\theta = \frac{1}{2}\pi + \bar{\theta}$; $\phi = \frac{1}{2}\pi - \bar{\phi}$. In order to make the velocity components positive we introduce $\bar{u} = u$, $\bar{v} = v$, $\bar{w} = -w$.

For the steady conical flow $\partial/\partial t$ and $\partial/\partial r$ vanish and the equations of motion are

$$\left.\begin{aligned}
&\frac{\partial}{\partial\theta}(\rho v\cos\theta) + \frac{\partial}{\partial\phi}(\rho w) + \partial\rho u\cos\theta = 0,\\
&v\cos\theta\frac{\partial u}{\partial\theta} + w\frac{\partial u}{\partial\phi} - (v^2+w^2)\cos\theta = 0,\\
&\rho v\cos\frac{\partial v}{\partial\theta} + \rho w\frac{\partial v}{\partial\phi} + \rho uv\cos\theta + \rho w^2\sin\theta = -\cos\theta\frac{\partial p}{\partial\theta},\\
&\rho v\cos\theta\frac{\partial w}{\partial\theta} + \rho w\frac{\partial w}{\partial\phi} + \rho uw\cos\theta - \rho vw\sin\theta = -\frac{\partial p}{\partial\phi},\\
&\left(r\cos\theta\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial\phi}\right)(h + \tfrac{1}{2}(u^2+v^2+w^2)) = 0.
\end{aligned}\right\} \qquad (2.1\,a\text{–}e)$$

Following Dorodnitcyn (1959) we put these equations in divergence form. Equation 2.1(a) is already in that form.

The equations could then be generalized as

$$\frac{\partial}{\partial\theta}(P_i\cos\theta) + \frac{\partial Q_i}{\partial\phi} + L_i\cos\theta + G_i\sin\theta = 0 \quad (i = 1, 2, \ldots, 5), \qquad (2.2)$$

where

$$\begin{array}{llll}
P_1 = \rho v, & Q_1 = \rho w, & L_1 = 2\rho u, & G_1 = 0;\\
P_2 = \rho uv, & Q_2 = \rho uw, & L_2 = \rho(2u^2 - v^2 - w^2), & G_2 = 0;\\
P_3 = p + \rho v^2, & Q_3 = \rho vw, & L_3 = 3\rho uv, & G_3 = p + \rho w^2;\\
P_4 = \rho vw, & Q_4 = p + \rho w^2, & L_4 = 3\rho uw, & G_4 = -\rho vw;\\
P_5 = \rho uH, & Q_5 = \rho wH, & L_5 = 2\rho H, & G_5 = 0;
\end{array}$$

and

$$H = h + \tfrac{1}{2}(u^2 + v^2 + w^2).$$

We shall leave the energy equation ($i = 5$) in differential form and replace it later by $H =$ constant, which is the integral of this equation for inviscid adiabatic gas. Since the enthalpy 'h' can be considered as a function of the pressure and density, equations (2.2) comprise of a system of five equations for the five unknowns u, v, w, p and ρ.

Boundary conditions

The flow field of interest is bounded by the wing surface on one side and by the shock surface on the other side. The boundary conditions are therefore specified along these two surfaces. The boundary condition on the body is simply the tangency condition, i.e.

$$v = 0 \quad \text{for} \quad \theta = 0 \quad \text{(for flat delta wing).} \tag{2.3}$$

The boundary conditions on the shock surface express the conservation of mass, momentum and energy across the shock wave discontinuity. Their formulation necessitates the introduction of two new quantities ϵ and σ. ϵ is defined as the angle between the wing and the shock in the plane ϕ = constant, so that the shock surface is given by $\theta_s = \epsilon$. Clearly ϵ is a function of ϕ. σ is measured in planes which are perpendicular to both the wing surface and the planes ϕ = constant. It is the angle between the line tangent to the trace of the shock and the trace of the wing in that plane. From geometrical considerations

$$d\epsilon/d\phi = -\tan\sigma\cos\epsilon. \tag{2.4}$$

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the unit vectors in the $(\bar{x}, \bar{y}, \bar{z})$ directions. The upstream vector is then given as

$$\mathbf{q}_\infty = -U_\infty \sin\alpha\mathbf{j} + U_\infty \cos\alpha\,\mathbf{k}. \tag{2.5}$$

In order to write down the shock relations, we must determine the velocity components tangential and normal to the shock. This is accomplished in two stages. First, the unit vectors along (r, θ, ϕ) directions $(\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})$ are expressed in terms of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

$$\left.\begin{aligned} \hat{\mathbf{l}} &= \cos\theta\sin\phi\mathbf{i} + \sin\theta\mathbf{j} + \cos\theta\cos\phi\mathbf{k}, \\ \hat{\mathbf{m}} &= \sin\theta\sin\phi\mathbf{i} + \cos\theta\mathbf{j} - \sin\theta\cos\phi\mathbf{k}, \\ \hat{\mathbf{n}} &= \cos\theta\mathbf{i} - \sin\phi\mathbf{k}. \end{aligned}\right\} \tag{2.6}$$

Next introduce the co-ordinate system (r', θ', ϕ') where (r', ϕ') directions lie in the surface of the shock, while the θ' direction is normal to it. The respective unit vectors in these new directions are given by

$$\left.\begin{aligned} \mathbf{l} &= \hat{\mathbf{l}}, \\ \mathbf{m} &= \cos\sigma\hat{\mathbf{m}} + \sin\sigma\hat{\mathbf{n}}, \\ \mathbf{n} &= -\sin\sigma\hat{\mathbf{m}} + \cos\sigma\hat{\mathbf{n}}. \end{aligned}\right\} \tag{2.7}$$

The components of the velocity $\mathbf{q}_\infty$ upstream of the shock

$$\mathbf{q}_\infty = q_\infty r'\mathbf{l} + q_\infty\theta'\mathbf{m} + q_\infty\phi'\mathbf{n} \tag{2.8}$$

are found at the shock $0 = \epsilon$ by scalar multiplication of the vector and the three unit vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$;

$$\begin{aligned} q_\infty r' &= U_\infty[\cos\alpha\cos\epsilon\cos\phi - \sin\alpha\sin\epsilon], \\ q_\infty\theta' &= -U_\infty[\cos\alpha\,(\cos\phi\sin\epsilon\cos\sigma + \sin\phi\sin\sigma) + \sin\alpha\cos\sigma\cos\epsilon], \\ q_\infty\phi' &= U_\infty[\cos\alpha\,(\cos\phi\sin\sigma\sin\epsilon - \sin\phi\cos\sigma) + \sin\alpha\sin\sigma\cos\epsilon]. \end{aligned} \tag{2.9}$$

Two of these three velocity components lie in the plane tangential to the shock surface, i.e. $(q_\infty r')$ and $(q_\infty\phi')$ while $(q_\infty\theta')$ is normal to the shock surface. Consequently

$$\left.\begin{aligned} (\mathbf{q}_T)_\infty &= (q_\infty r')\,\mathbf{l} + (q_\infty\phi')\,\mathbf{n}, \\ (\mathbf{q}_N)_\infty &= (q_\infty\theta')\,\mathbf{m}. \end{aligned}\right\} \tag{2.10}$$

$(\mathbf{q}_T)$, $(\mathbf{q}_N)$ are respectively velocities tangential and normal to the shock surface.

The Rankine–Hugoniot relations at the shock can be written as

$$\left.\begin{aligned} (q_T)_s &= (q_T)_\infty, \\ p_s - p_\infty &= \rho_\infty (q_N)_\infty^2 (1-\rho_\infty/\rho_s), \\ (q_N)_s &= (q_N)_\infty \rho_\infty/\rho_s, \\ h_s - h_\infty &= \tfrac{1}{2}(q_N)_\infty^2(1-(\rho_\infty/\rho_s)^2). \end{aligned}\right\} \tag{2.11}$$

The subscripts s and ∞ denote respectively quantities downstream and upstream of the shock. Having found $(q_T)_s$ and $(q_N)_s$ we must resolve them along (r, θ, ρ) directions in order to find u_s, v_s, w_s, i.e.

$$\left.\begin{aligned} u_s &= [(\mathbf{q}_T)_s + (\mathbf{q}_N)_s].\hat{\mathbf{1}}, \\ v_s &= (\rho_\infty/\rho_s)\,(q'_\phi)_\infty \cos\sigma - (q'_\phi)_\infty \sin\sigma, \\ w_s &= (\rho_\infty/\rho_s)\,(q'_\theta)_\infty \sin\sigma - (q'_\phi)_\infty \cos\sigma. \end{aligned}\right\} \tag{2.12}$$

For a perfect gas $h = (\gamma/(\gamma-1))p/\rho$, where γ is the ratio of specific heats, and hence it is possible to get the shock relations in closed form. In this case the pressure and density ratios across the shock are functions of two parameters only, γ and the Mach number normal to the shock M_N given by

$$M_N = |q_N|_\infty/\alpha_\infty, \tag{2.13}$$

where $\qquad \alpha_\infty^2 = \gamma p_\infty/\rho_\infty$, the upstream sound speed.

From (2.9) and (2.10) we can substitute for $(q_N)_\infty$ in (2.13) to obtain

$$M_N = M_\infty\{\cos\alpha(\cos\phi\cos\sigma\sin\epsilon + \sin\phi\sin\sigma) + \sin\alpha\cos\sigma\cos\epsilon\}, \tag{2.14}$$

where M_∞ is the upstream Mach number; in terms of which

$$\begin{aligned} &\frac{p_s}{p_\infty} = \frac{2\gamma M_N^2 - (\gamma-1)}{\gamma+1}, \quad \frac{\rho_s}{\rho_\infty} = \frac{(\gamma+1)M_N^2}{2+(\gamma-1)M_N^2}, \\ &\frac{u_s}{U_\infty} = \cos\alpha\cos\phi\cos\epsilon - \sin\alpha\sin\epsilon, \\ &\frac{v_s}{U_\infty} = -\frac{\sin\sigma(2+(\gamma-1)M_N^2)}{(\gamma+1)M_\infty M_N} \\ &\qquad + \cos\sigma\{\cos\alpha(\sin\sigma\cos\phi\sin\epsilon - \sin\phi\cos\sigma) + \sin\alpha\sin\sigma\cos\epsilon\}, \\ &\frac{w_s}{U_\infty} = -\frac{\sin\sigma(2+(\gamma-1)M_N^2)}{(\gamma+1)M_\infty M_N} \\ &\qquad + \cos\sigma\{\cos\alpha(\sin\sigma\cos\phi\sin\epsilon - \sin\phi\cos\sigma) + \sin\alpha\sin\sigma\cos\epsilon\}. \end{aligned} \tag{2.15}$$

This completes the formulation of the boundary conditions at the shock surface.

Now we have six differential equations and six boundary conditions, but the total number of dependent variables is seven, u, v, w, p, ϕ, ϵ and σ. Since the unknown angle σ is introduced through the boundary conditions, the situation is rendered determinate by specifying an additional boundary condition $\sigma(0) = 0$ which expresses the fact that in the plane of symmetry ($\phi = 0$) of the wing the shock is parallel to the wing surface.

3. Reduction to ordinary differential equations

Before applying the one-strip approximation in Dorodnitcyn's method of integral relations, we shall give a brief account of the general formulation of this method.

The equations of motion are already in divergence form. The shock layer is divided into N strips, so that the edge of the jth strip is given by

$$\theta_j = \zeta_j \epsilon(\phi); \quad \zeta = (N-j+1)/N \quad (j = 1, 2, \ldots, N).$$

On the shock $\theta = \epsilon(\phi)$ hence $j = 1$ describes the shock. On the body $\phi = 0$ so that we will define $\zeta_0 = 0$ and use the superscript 0 to define quantities on the body surface. Similarly, quantities along the jth curve will be denoted by the superscript 'j'.

We integrate from $\theta = 0$ to $\theta = \theta_j$ to obtain

$$P_1^j \cos\theta - P_i^0 + \frac{d}{d\phi}\int_0^{\theta_j} Q_i \, d\theta - Q_i^j \frac{d\theta_j}{d\phi} + \int_0^{\theta_j} (L_i \cos\theta + G_i \sin\theta)\, d\theta = 0$$
$$(i = 1, 2, \ldots, 5; \; j = 1, 2, \ldots, N). \quad (3.1)$$

The system of equations (3.1) comprises $5N$ equations which together with the boundary condition, differential equation (2.4), constitute a system of $5N+1$ equations. The number of unknowns [u, v, w, p, ϕ on the edges of $N-1$ strips, that is $5(N-1)$, ϵ, σ and u^0, w^0, p^0, ρ^0] is $5N+1$ so the problem is determinate.

In order to reduce the system to ordinary differential equations, the Q_i, $L_i \cos\theta$, $G_i \sin\theta$ functions are represented by polynomials in ζ where $\zeta = \theta/\epsilon(\phi)$ is the normalized variable. Consequently

$$\left.\begin{aligned} Q &= \sum_{m=0}^{N} \alpha_m(\phi)\,\zeta^m, \\ L\cos\theta &= \sum_{m=0}^{N} b_m(\phi)\,\zeta^m, \\ G\sin\theta &= \sum_{m=0}^{N} c_m(\phi)\,\zeta^m, \end{aligned}\right\} \quad (3.2)$$

where the α_m's, b_m's, c_m's are functions of u, v, w, p, ρ at the edges of the mth strip, i.e.

$$\alpha_m(\phi) = \alpha_m(u^m, v^m, w^m, p^m, \rho^m, m, \epsilon). \quad (3.3)$$

When (3.2) are put in (3.1) we obtain a system of ordinary differential equations for u, v, w, etc., on the edge of each strip. The solution is then obtained by means of numerical integration along each strip from $\phi = 0$ to any plane ϕ = constant.

Case $N = 1$

If we let X stand for either Q_i, $L_i \cos\theta$, $G_i \sin\theta$ then for the case of $N = 1$

$$X_i = X_i^0 - \zeta^m (X_i' - X_i^0), \quad (3.4)$$

where $m = 1$ if X_i contains v and $m = 2$ if X_i does not contain v^2. This assures us that not only the variables but also their first derivatives are matched at the wing surface, since it can be shown that $\partial p/\partial\theta$, $\partial \rho/\partial\theta$, $\partial u/\partial\theta$, $\partial w/\partial\theta$ all vanish at the wing surface $\theta = 0$. This implies that there is an infinitesimal layer across the wing surface across which entropy is constant.

The unknowns in the case of $N = 1$ are u^0, w^0, p^0, ρ^0, σ and ϵ. It should be noted that $v^0 = 0$ is known from the boundary condition of the body.

Equation (3.1) for the case $N = 1$ reduces to

$$P_i^0 \cos\theta - P_i^0 + \frac{d}{d\phi}\int_0^1 \epsilon Q_i d\zeta - Q_i^1 \frac{d\epsilon}{d\rho} + \epsilon\int_0^1 L_i \cos(\epsilon\zeta)\, d\zeta + \epsilon\int_0^1 G_i \sin(\epsilon\zeta)\, d\zeta = 0$$
$$(i = 1, 2, 3, 4, 5), \quad (3.5)$$

where θ has been replaced by ζ from the relation $\theta = \epsilon\zeta$. The last of equation (3.5) is replaced by an algebraic equation

$$\frac{\gamma}{\gamma-1}\frac{p^0}{\rho_0} + \tfrac{1}{2}(u^{02} + w^{02}) = H = \text{constant}. \quad (3.6)$$

This implies that the total enthalpy of the flow is constant throughout.

When equations (3.4) are put in (3.5) we obtain equations (A 1) in the appendix. These together with the geometric equation (2.4) are solved to give

$$\left.\begin{aligned} d\epsilon/d\phi &= \tan\sigma\cos\epsilon, \\ d\sigma/d\phi &= 3R_3/\epsilon K_4, \\ du^0/d\phi &= (R_6 - u^0R_5)/\tfrac{2}{3}\epsilon w^0\rho^0, \\ dw^0/d\phi &= (R_5/\tfrac{2}{3}\epsilon\rho^0) - (3K/10\epsilon\rho^0(g - w^{02})), \\ d\rho^0/d\phi &= 3K/10\epsilon w^0(g - w^{02}), \\ \text{where} \qquad g &= \gamma p^0/\rho^0. \end{aligned}\right\} \quad (3.7)$$

Expressions for R_s are written out in the appendix. We have chosen as our reference quantities p_∞, ρ_∞, T_∞ so that U_∞ is replaced by $\gamma^{\frac{1}{2}} m_\infty$ where m_∞ is the upstream Mach number and H is then given by

$$H = \gamma(\gamma - 1) + \tfrac{1}{2}\gamma m_\infty^2 \quad (m_\infty = M_\infty).$$

4. Boundary-value problem

Let ϕ_e be the leading edge angle of the wing. At

$$\phi = \phi_e, \quad \epsilon = v^1 = 0. \quad (4.1)$$

From (2.15) we obtain

$$\begin{aligned} &\tan^3\sigma(\gamma+1)\, m_\infty^2 \cos\alpha\sin\phi_e \sin\alpha \\ &\quad + \tan^2\sigma\{(\gamma-1)\, m_\infty^2 \sin^2\phi_e\cos^2\alpha + (\gamma+1)\, m_\infty^2(\sin^2\alpha - \sin^2\phi_e\cos^2\alpha) + 2\} \\ &\quad + \tan\sigma\{2(\gamma-1)\, m_\infty^2 \sin\alpha\cos\alpha\sin\phi_0 - (\gamma+1)\, m_\infty^2 \sin\alpha\cos\alpha\sin\phi_e\} \\ &\quad + \{2 + (\gamma-1)\, m_\infty^2 \sin^2\alpha\} = 0. \end{aligned} \quad (4.2)$$

This is a cubic equation for $\tan\sigma$. For a particular set of Mach number, angle of attack, sweepback angle and γ, the cubic is solved iteratively on the computer to obtain the shock angle σ at the leading edge. This iterative solution gives us the weak shock which is the physically applicable of the three solutions. When σ is known at the leading edge, the other flow variables could be found.

The flow near the leading edge generated by the plane oblique shock is uniform. This flow terminates on the singular Mach cone.

We shall now look closely at the singularities of our system. From (3.7), we see that there are singularities at $w^0 = 0$ and $w^{02} = g = \gamma p^0/\rho^0$. The first occurs at $\phi = 0$ (root chord).

This is a fixed singularity which leads to the establishment of a regularity condition there. At this point $\sigma = 0$ and from symmetry considerations $d\rho^0/d\phi = 0$ which also leads to $du^0/d\phi = 0$.

The second singularity occurs at the sonic point $\phi = \phi_s$. This is a regular movable singularity inboard of the wing which depends on our parameters. Near the sonic point, the velocity component increases from subsonic to supersonic in the positive ϕ direction. Thus $dw^0/d\phi$ must be positive for positive w^0 (outgoing cross flow) and negative for negative w^0 (incoming cross flow). This singularity could be seen to be of the saddle point type.

The third of (3.7) could be written as

$$dw^0/d\phi = -u^0 + 3F/[2\epsilon\rho^0(w^{02} - g)], \tag{4.3}$$

where F is a function of ϕ, ϵ, σ, ρ_0, u^0, v^0, w^0. At the sonic point, the denominator of the second term vanished. For the velocity gradient to be finite the numerator is made equal to zero. Hence we obtain

$$\lim_{\phi = \phi_s} (dw^0/d\phi) = \bar{u}^0 + \Delta$$

where Δ is a finite parameter and $\bar{u}^0$ is the value of u^0 at ϕ_s. With this, we also obtain

$$\lim_{\phi \to \phi_s} \frac{d\rho^0}{d\phi} = \frac{3R_4}{2\epsilon\bar{w}^0} - \frac{\rho^0}{\bar{w}^0}(\Delta - \bar{u}^0)$$

(all quantities evaluated at the sonic singular point).

The numerical integration of our equations (ordinary differential) in the non-uniform domain is started from the sonic point. Varying Δ (starting from zero say), series of w^0 are generated from the sonic point and only one of them passes through the origin (i.e satisfies the boundary condition $w^0 = 0$ at $\phi = 0$).

The standard boundary-value technique is used to find this value of Δ. We try Δ_1 to obtain w_1, at $\phi = 0$, then Δ_2 to obtain w_2 and we then try Δ_3 where $\Delta_3 = (\Delta w_2 - \Delta_2 w_1)/(w_2 - w_1)$ to yield w_3. The calculations are repeated with Δ_2 and Δ_3 and the whole process is repeated until $w^0 = 0$ at $\phi = 0$.

We use a special fourth-order Runge–Kutta–Merson procedure for the integration. The procedure requires that we know both the flow fields as well as their derivatives at the starting point if there is a singularity there. Otherwise, we feed in only the flow variables. The step length is variable and integration is performed inwards.

Our computations were carried out for wings whose flow Babaev had already calculated so that we can compare the two results. The results are plotted in figures 3–6.

For M_∞ very large (1000 say) some of the flow field gradients were found to be very large (because of the large values of the fields themselves). We changed our dimensionalization scheme by replacing ρ_{ref}, U_{ref}, p_{ref}, by ρ_∞, U_∞, $\rho_\infty u_\infty^2$ respectively.

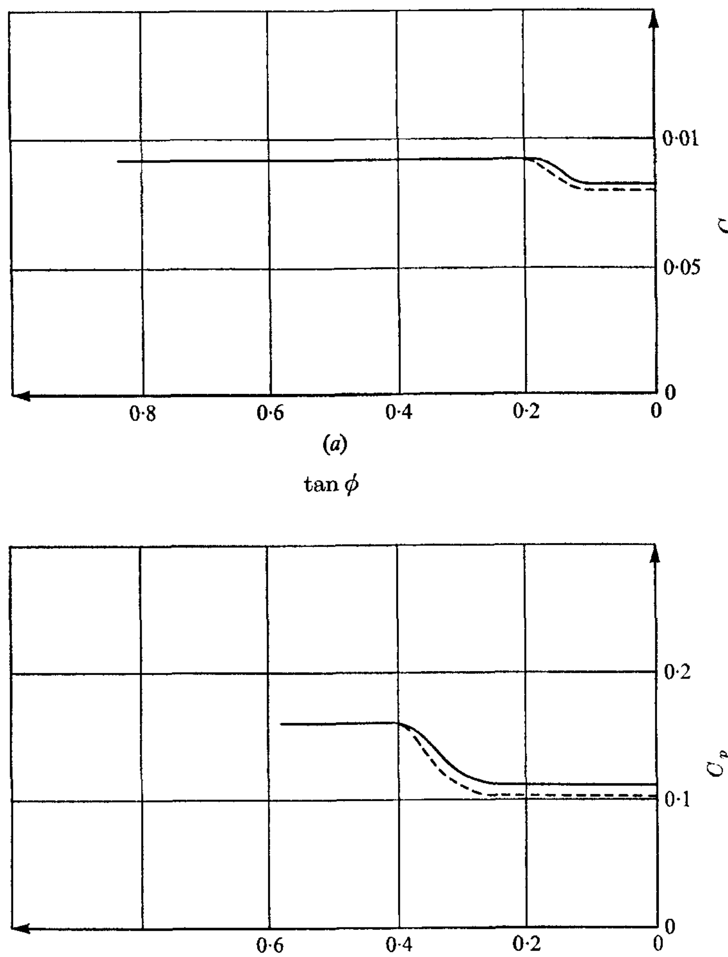
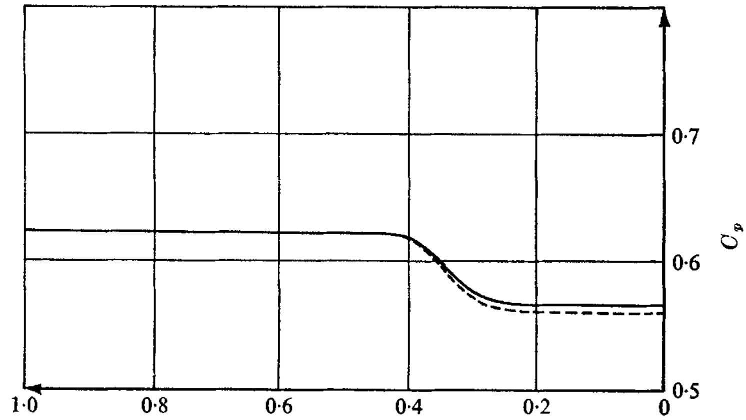


FIGURE 3. Pressure distributions. – – –, Babaev. (a) $M_\infty = 6$, $\alpha = 9°$, $\chi = 50°$. (b) $M_\infty = 4$, $\alpha = 10°$, $\chi = 60°$.

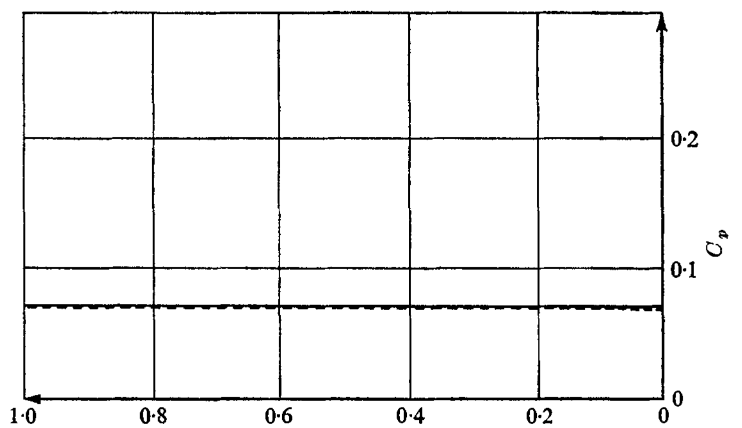
The results show that u^0 increases very slightly towards the centreline, which confirms the basis for hypersonic small disturbance theory ($u = u_0 + \tau^2 u_i + \ldots$, where τ is slenderness parameter).

w^0 and σ decrease steadily and attain zero at $\phi = 0$ while ϵ increases and attains its maximum at $\phi = 0$. Both pressure and density decrease as the root chord is approached.



(a)

tan ϕ



(b)

tan ϕ

FIGURE 4. Pressure distributions. – – –, Babaev. (a) $M_\infty = 1000$, $\alpha = 30°$, $\chi = 45°$. (b) $M_\infty = 1000$, $\alpha = 10°$, $\chi = 45°$.

5. Conclusion

We have established that the method of integral relations applied to the problem of calculating the inviscid hypersonic flow past the lower surface of a delta wing yields results which agree very closely with those obtained by Babaev using the finite difference scheme and the method of successive iterations to the flow fields.

Our results contradict the assertion by Brook (1965) that the method with one strip approximation leads, for the mixed problem to an undetermined solution.

Our work supports Babaev's results that the transition between conically supersonic and subsonic flow takes place without the shock wave which Brook (1965) and Bulakh (1961) have suggested was necessary.

The 'convergence' of our solution has not been proved. We rely only on the good agreement with Babaev's calculations. It would have been better to carry out the calculations for $N = 2$ and compare the solutions but the complexity of our equations makes this very difficult. Finally, we remark that the method which has been developed in this investigation provides a suitable means for calculating approximately a class of flows of considerable aerodynamic importance.

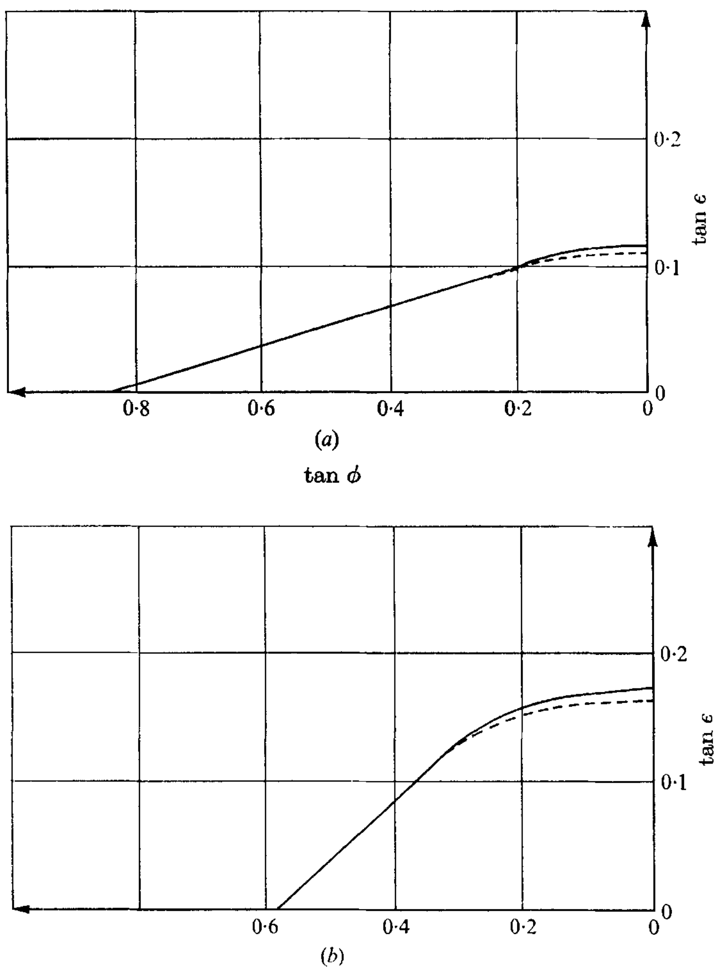


FIGURE 5. Shock shapes. – – –, Babaev. (a) $M_\infty = 6$, $\alpha = 9°$, $\chi = 50°$. (b) $M_\infty = 4$, $\alpha = 10°$, $\chi = 60°$.

Appendix

The equations referred to in § 3 are

$$\tfrac{2}{3}\epsilon\rho^0 \frac{dw^0}{d\phi} + \tfrac{2}{3}\epsilon w^0 \frac{d\rho^0}{d\phi} = R_5 \ldots,$$

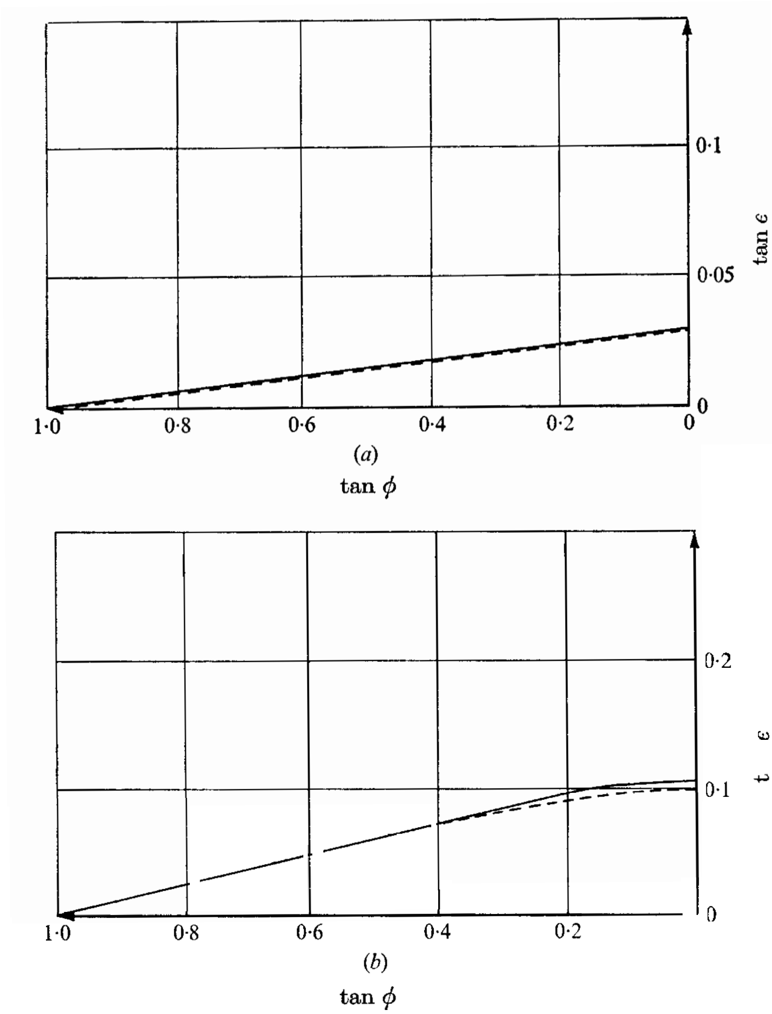


FIGURE 6. Shock shapes. – – –, Babaev. (a) $M_\infty = 1000$, $\alpha = 10°$, $\chi = 45°$, (b) $M_\infty = 1000$, $\alpha = 30°$, $\chi = 45°$.

$$\left.\begin{aligned}
\tfrac{2}{3}\epsilon\rho^0 w^0\frac{du^0}{d\phi}+\tfrac{2}{3}\epsilon\rho^0 u^0\frac{dw^0}{d\phi}+\tfrac{2}{3}\epsilon u^0 w^0\frac{d\rho^0}{d\phi} &= R_6\ldots,\\
\tfrac{1}{3}\epsilon\, K_3\frac{d\sigma}{d\phi} &= R_3\ldots,\\
\tfrac{4}{3}\epsilon\rho^0 w^0\frac{dw^0}{d\phi}+\tfrac{2}{3}\epsilon w^{02}\frac{d\rho^0}{d\phi}+\tfrac{2}{3}\epsilon\,\frac{dp^0}{d\phi} &= R_7\ldots,\\
\tfrac{4}{21}\epsilon u^0\rho^0\frac{du^0}{d\phi}+\tfrac{4}{21}\epsilon w^0\rho^0\frac{dw^0}{d\phi}-\tfrac{2}{3}\epsilon\,\frac{p^0}{\rho^0}\frac{d\rho^0}{d\phi}+\frac{2\epsilon}{3}\frac{dp^0}{d\phi} &= 0,
\end{aligned}\right\}\quad \text{(A 1)}$$

where $$R_5 = R_1+\frac{K_1R_3}{K_3},\quad R_6 = R_2+\frac{K_2R_3}{K_3},\quad R_7 = R_4+\frac{K_4R_3}{K_3},$$

$$K_1 = bw'zm_\infty - \rho'v' - \rho'(bfU_\infty \sin\sigma M_\infty + U_\infty \cos\sigma(M_N/M_\infty)),$$

$$K_2 = u'K_1,$$

$$K_3 = \tfrac{3}{2}v'K_1 + \tfrac{3}{2}\rho'w'(w' - bfM_\infty U_\infty \cos\sigma + U_\infty(M_N/M_\infty)\sin\sigma),$$

$$K_4 = -2\rho'w'(U_\infty bfM_\infty \sin\sigma v' + U_\infty(M_N/M_\infty)\cos\sigma) + bM_\infty(\alpha d + zw'^2).$$

$$\alpha = \frac{2+(\gamma-1)M_N^2}{(\gamma-1)M_\infty M_N},$$

$$b = \cos\alpha(\cos\phi\sin\sigma\sin\epsilon - \sin\phi\cos\sigma) + \sin\alpha\sin\sigma\cos\epsilon;$$

$$c = \cos\alpha\cos\phi\cos\epsilon - \sin\alpha\sin\epsilon,$$

$$\alpha\alpha = -\cos\alpha(\cos\phi\cos\sigma + \sin\sigma\sin\phi\sin\epsilon),$$

$$\alpha c = \cos\alpha(\sin\phi\sin\epsilon\cos\sigma - \sin\sigma\cos\phi),$$

$$\alpha d = \frac{4\gamma M_N}{\gamma+1}, \quad f = \frac{(\gamma-1)M_N^2 - 2}{(\gamma+1)M_\infty M_N^2},$$

$$z = \frac{4}{(\gamma+1)M_N M_\infty^2 \alpha^2}.$$

$$\begin{aligned} R_1 = {} & \tfrac{1}{3}\epsilon(\alpha czw'M_\infty - \rho'U_\infty(\alpha cfM_\infty\sin\sigma + \alpha\alpha\cos\sigma)) - \tfrac{4}{3}\epsilon\rho^0u^0 \\ & + \tfrac{2}{3}\tan\sigma\cos\epsilon(\rho^0w^0 - \rho'w') - \rho'\cos\epsilon(v' + \tfrac{2}{3}\epsilon u') \\ & + \epsilon\tfrac{1}{3}c\sin\sigma\cos\epsilon(w'zM_\infty - \rho'U_\infty\sin\sigma(fM_\infty - 1)), \end{aligned}$$

$$\begin{aligned} R_2 = {} & \tfrac{1}{3}\epsilon\{\alpha czu'w'M_\infty + \rho'w'U_\infty\cos\alpha\sin\phi\cos\epsilon - \rho'u'U_\infty(\alpha\alpha\cos\sigma + \alpha cfM_\infty\sin\sigma)\} \\ & - \tfrac{1}{3}\epsilon\rho'w'\alpha b\tan\sigma\cos\epsilon - \tfrac{2}{3}\tan\sigma\cos\epsilon(\rho'u'w' - \rho^0u^0w^0) + \tfrac{2}{3}\epsilon\rho^0(w^{02} - 2u^{02}) \\ & + \tfrac{1}{3}\epsilon c\sin\sigma C_n\epsilon u'(w'zM_\infty - \rho'U_\infty\sin\sigma(fM_\infty - 1)) - \rho'u'v'\cos\epsilon \\ & - \tfrac{1}{3}\epsilon\rho'\cos\epsilon(2u'^2 - v'^2 - w'^2), \end{aligned}$$

$$\begin{aligned} R_3 = {} & \cos\epsilon(p' + \rho'v'^2) - p^0 + \tfrac{1}{3}\epsilon\sin\epsilon(p' + \rho w'^2) \\ & + \tfrac{3}{2}\epsilon\rho'u'v'\cos\epsilon + \tan\sigma\cos\epsilon\rho'v'w' + \tfrac{1}{2}\epsilon\tan\sigma\cos\epsilon \\ & \times\{-v'w'zc\cos\sigma M_\infty + \rho'v'U_\infty c\sin\sigma\cos\sigma(fM_\infty - 1) \\ & + \rho'w'cU_\infty(\sin^2\sigma + fM_\infty\cos^2\sigma)\} + \tfrac{1}{2}\epsilon\{\rho'v'U_\infty(\alpha\alpha\cos\sigma + \alpha c\sin\sigma fM_\infty) \\ & - \alpha czv'w'M_\infty + \rho'w'U_\infty(-\alpha\alpha\sin\sigma + \alpha cfM_\infty\cos\sigma)\}, \end{aligned}$$

$$\begin{aligned} R_4 = {} & 2\rho^0u^0w^0 - \rho'v'w' - \epsilon\cos\epsilon\rho'u'w' + \tfrac{1}{2}\epsilon\sin\epsilon\rho'v'w' \\ & + \tfrac{1}{3}\alpha c\epsilon M_\infty(\alpha d + zw'^2) - \tfrac{2}{3}\epsilon\rho'w'U_\infty(\alpha\alpha\cos\sigma + \alpha cfM_\infty\sin\sigma) \\ & + \tfrac{1}{3}c\epsilon\sin\sigma\cos\epsilon\{M_\infty(\alpha d + zw'^2) - 2\rho'w'U_\infty\sin\sigma(fM_\infty - 1)\} \\ & + \tfrac{2}{3}\tan\sigma\cos\epsilon\{(p^0 + \rho^0w^{02}) - (p' + \phi'w'^2)\}, \end{aligned}$$

$$K = 7w^0R_7 - 12w^{02}R_5 + 2u^0(R_6 - u^0R_5).$$

The simultaneous equations (A 1) are solved to obtain the gradients written out in system (3.7).

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